The Fundamental Theorem of Calculus
Using the Rule of Three

A. Approximations with Riemann sums.

The area under a curve can be approximated through the use of Riemann (or rectangular) sums:

\[ \text{Area} \approx \sum_{k=1}^{n} f(x_k) \Delta x_k \]

Let \( L_n = \) Sum of \( n \) rectangles using the left-hand x-coordinate of each interval to find the height of the rectangle.

Let \( R_n = \) Sum of \( n \) rectangles using the right-hand x-coordinate of each interval to find the height of the rectangle.

Let \( M_n = \) Sum of \( n \) rectangles using the midpoint x-value of each interval to find the height of the rectangle.

1. Let \( f(x) = 9 - x^2 \) on \([0,3] \)
   
a) If all intervals are the same width are there are six rectangles, what is the value of \( \Delta x \)? \( \frac{1}{2} \). Use this value to compute the following.

Draw a sketch of the function along with the indicated rectangles.

Use your calculators with the Lists in the STAT section to compute the following Riemann sums. (Commands for the TI-83/84 are shown)
List1 = x-coordinate for each rectangle.
List2 = y-coordinate for each rectangle (using formula in title bar).
List3 = width of interval.
List4 = Area of each rectangle (using formula in title bar).
From the home screen 2nd LIST, MATH, SUM(L4) will sum the values in List 4.

\[ \frac{1}{2} \sum_{c=0}^{5} (9 - (\frac{c}{2})^2) = 20.125 \]

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b) Suppose \( \Delta x_k \) has width equal 0.5. Choose a random value for \( x_k \) in each interval and compute the area of each Riemann rectangle. Find the total of the rectangles.

c) Choose six randomly sized intervals and choose a point at random in each interval. Compute the area of the Riemann rectangle and find the total area.
Example 2. Let \( f(x) = 3^x \) on \([-1, 3]\)

a) After sketching the graph and rectangles, find
\[
L_8 = 5 \left( 3^{-1} + 3^{-0.5} + 3^0 + 3^{0.5} + 3^1 + 3^{1.5} + 3^{2} + 3^{2.5} \right) = 36.42712 = 18.214
\]
\[
R_8 = 5 \left( 3^{-0.5} + 3^0 + 3^{0.5} + 3^1 + 3^{1.5} + 3^{2} + 3^{2.5} + 3^{3} \right) = 31.547
\]
\[
M_8 = 5 \left( 3^{-0.75} + 3^{-0.25} + 3^{0.25} + 3^{1.75} + 3^{1.25} + 3^{2.25} + 3^{2.75} \right) = 23.971
\]

b) Suppose \( \Delta x_k \) has width equal 0.5. Choose a random value for \( x_k \) in each interval and compute the area of each Riemann rectangle. Find the total area approximated by the rectangles.

c) Choose eight randomly sized intervals and choose a point at random in each interval. Compute the area of the Riemann rectangle and find the total area approximated by the rectangles.

So far, we haven’t applied any calculus. We’ve only done geometry. Calculus always requires the use of a limit. The definite integral is simply the limit of a Riemann sum as the width of the interval gets smaller and smaller as indicated in the formula below.

\[
\lim_{\Delta x \to 0} \sum_{k=1}^{\infty} f(x_k) \Delta x_k = \int_{a}^{b} f(x) \, dx
\]

The \texttt{fnInt} option on your calculator uses Riemann sums with \( \Delta x = 0.0001 \). It computes the area of each rectangle and gives the total of all rectangles from \( x = a \) to \( x = b \).

Sketch a quick graph and shade the indicated region of each of the following functions. Use \texttt{fnInt} to approximate the value of each definite integral.

3. \( \int_{0}^{2} x^2 \, dx \)

4. \( \int_{0}^{-2} x^2 \, dx \)

5. \( \int_{-2}^{0} x^2 \, dx \)

Why is the answer to problem 4 a negative number?
Draw a sketch for each problem and use your calculator to find the answer. (Make sure your calculator is set to RADIAN mode for these problems.)

6. \[ \int_{0}^{2} \sin x \, dx = \]

7. \[ \int_{\pi}^{2\pi} \sin x \, dx = -2 \]

8. \[ \int_{0}^{2\pi} \sin x \, dx = 0 \]

9. \[ \int_{\pi}^{0} \sin x \, dx = -2 \]

10. \[ \int_{2\pi}^{\pi} \sin x \, dx = 2 \]

11. \[ \int_{0}^{-\pi} \sin x \, dx = 2 \]

Give a geometric explanation why some of the answers are negative and some are zero.

Below x-axis moving in + direction is -
Below x-axis moving in - direction is +
Above x-axis moving in - direction is -

Use symbolic notation to write \( \int_{a}^{b} f(x) \, dx \)

\[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]

\[ \int_{a}^{b} f(x) \, dx = F(x) \Big|_{a}^{b} = F(b) - F(a) \]

\[ F'(x) = f(x) \quad F(x) \text{ is antiderivative} \]
B. Using the Rule of Three to verify the Fundamental Theorem, Part I.

**Conjecture:** What is the relationship between the “area function” \( A(x) \) and the original function \( f(t) \)?

**Example #1**

Given \( f(t) = 5 \) on \([0, x]\). We will find an area function, which we will call \( A(x) \), which represents the area under the graph of \( f(t) \) from \( t = 0 \) to \( t = x \) for different values of \( x \).

a. Fill in the chart below by evaluating \( A(x) = \int_0^x 5 \, dt \)

   Use \textbf{fnInt} on your calculator for the values of \( x \) given in the table

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(x) )</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>-5</td>
</tr>
</tbody>
</table>

b. Use the STAT Edit feature of your calculator and put the \( x \) values in \( L_1 \) and the corresponding values of \( A(x) \) in \( L_2 \). Use the STATPLOT feature to make a scatter plot of the ordered pairs.

c. Find a function \( A(x) \) that fits the data points: \( A(x) = \underline{5x} \)

d. Make a sketch of the function \( y = f(t) \) and shade in the region on the given interval. Use a geometric formula to find the area of the shaded region.

   \( A(x) = \underline{5x} \)

e. Use the definite integral formula to evaluate \( A(x) \).

\[
\int_0^x 5 \, dt = 5 \int_0^x 1 \, dt = 5 \left[ t \right]_0^x = 5x - 0 = 5x
\]

On another sheet of paper, complete steps a-e for each of the following functions over the given intervals. Write your answer in the space provided.

2. \( f(t) = 5 \) on \([1, x]\) \[\int_1^x 5 \, dt = 5 \left[ t \right]_1^x = 5(x) - 5(1) \frac{x}{5(x-1)} = 5(5) - 5(1) = 25 + 5 = 30\]

3. \( f(t) = 5 \) on \([-1, x]\) \[\int_{-1}^x 5 \, dt = 5 \left[ t \right]_{-1}^x = 5(5) - 5(-1) = 25 + 5 = 30\]

4. \( f(t) = t \) on \([1, x]\) \[\int_1^x t \, dt = \frac{1}{2} t^2 \left|_1^x \right. = \frac{1}{2}(x)^2 - \frac{1}{2}(1)^2 = \frac{1}{2}x^2 - \frac{1}{2}\]

5. \( f(t) = t \) on \([-1, x]\) \[\int_{-1}^x t \, dt = \frac{1}{2} t^2 \left|_{-1}^x \right. = \frac{1}{2}(x)^2 - \frac{1}{2}(-1)^2 = \frac{1}{2}x^2 - \frac{1}{2}\]

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Example #2.

Let \( f(t) = 2t - 2 \). Sketch the graph of \( f \) on \([-2,4]\).

Let \( A(x) = \int_0^x (2t-2) \, dt \).

Using your graph of \( f \), answer the following questions and sketch the graph of \( A \).
(Remember: The \textbf{y-value} of a point \((x,y)\) of function \( A \) represents the \textbf{area} in the graph of \( f \) from \( t = 0 \) to \( t = x \)).

Questions:

a. What is \( A(0) \)?

b. What is \( A(1) \)?

c. What is \( A(2) \)?

d. What is \( A(3) \)?

e. What is \( A(-1) \)?

f. What is \( A(-2) \)?

g. Where does \( A \) reach its minimum value?

h. What happens to \( A \) as \( x \to \infty \)? \( A(x) \to \infty \)

i. What happens to \( A \) as \( x \to -\infty \)? \( A(x) \to \infty \)

Plot the coordinates of the points found in parts a-f in the same window as the sketch of the graph of \( f(t) \).

Sketch what you think \( A(x) \) might look like.

\[
\int_0^x 2t-2 \, dt = \left. 2\frac{t^2}{2} - 2t \right|_0^x = x^2 - 2x
\]

\[
x^2 - 2x + 1 - 1 = (x-1)^2 - 1
\]
3. Verify your sketch by graphing $A(x)$ using $\text{fnInt}(2x - 2, x, 0, x)$ on your calculator.

4. "Guess" the equation for the function $y = A(x)$ that you sketched in question 2 and graphed in question 3. $A(x) = 5x$

5. Use Part I of The Fundamental Theorem to analytically derive the actual function.
   \[ \int_0^x 5 \, dt = 5t \bigg|_0^x = 5x - 5(0) = 5x \]

6. Is the answer to question 5 the same as your answer to question 4?
   \[ \text{Yes} \]

Answer the following questions about the relationship between the function $f(t)$ and $A(x) = \int_0^x f(t) \, dt$

1. What is the value of $A(a)$? 0

2. A maximum or minimum value of $A(x)$ occurs at a zero of $f$?
C. Fundamental Theorem, Part II

Functions can be written in a variety of ways. For example, \( f(x) = x^2 - 1 \) and \( f \rightarrow x^2 - 1 \) are two different ways to describe the same function.

Functions can also be described by integrals. Given: \( f(x) = \int_1^x 2t \ dt \), show that this function is the same as the other versions of \( f(x) \).

Conjecture: What is \( \frac{d}{dx} \left( \int_a^x f(t) \ dt \right) \)?

\[
F(t) = F(x) - F(a)
\]

a. Write \( \int_a^x f(t) \ dt \) in symbolic form.

b. Take the derivative the answer in (a) with respect to \( x \).

1. \[
\frac{d}{dx} \left( \int_2^x \sqrt{2 + \cos t} \ dt \right)
\]
\[
F'(t) \bigg|_a^x = \left( F(x) - F(a) \right)_x^a \Rightarrow F'(x) - 0 \Rightarrow F(x) = \sqrt{2 + \cos x}
\]

2. \[
\frac{d}{dx} \left( \int_2^x \sqrt{5 + t^3} \ dt \right)
\]
\[
F(t) \bigg|_a^x = \left[ F(x^2) - F(2) \right]_a^x \Rightarrow F'(x^2) \cdot 2x - 0 = f(x^2) \cdot 2x
\]

3. \[
\frac{d}{dx} \left( \int_2^{\sin x} \frac{3}{t + t^4} \ dt \right)
\]
\[
F(t) \bigg|_a^x = \left[ F(\sin x) - F(x^2 + x) \right]_a^x \Rightarrow \frac{d}{dx} \left[ F(\sin x) - F(x^2 + x) \right] = 2x \sqrt{5 + x^6}
\]
\[
f'(\sin x) \cos x - F'(x^2 + x) (2x + 1) = f(\sin x) \cos x - f(x^2 + x) \cdot (2x + 1) = \cos x \left( \frac{3}{(2x + 1)^3} \frac{\sin x \sin^4 x}{x^2 + x + (x^2 + x)^4} \right)
\]